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Title: FRACTAL GEOMETRY IN PHYSICS
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Work has proceeded smoothly in several directions. The principal achievements are as follows:

Part A: Simulation studies of DLA: coexistence of self similar features and of features that clearly depart from self similarity.

In a first approximation, DLA clusters are self-similar, i.e., their complication is about the same as all scales of observation sufficiently above the scale of the atoms. While clear-cut departures from self similarity have been known to exist, the Yale ONR project has extended them, and has been the first to place them within a more correct overall geometric picture of DLA. Our very careful computer studies use clusters of up to 100 million atoms. This is a world record and its achievement is to be credited to outstanding computer help by H. Kaufman.

A1) The topological (branching) properties of DLA are asymptotically self similar.

Two papers by Yekutieli, Mandelbrot and Kaufman (accepted by J. Physics A) examine the behavior of Horton-Strähler branching ratios.

It is found that low-order ratios reach a common value and cease to vary well before the clusters reach the maximum value we can study; higher ratios appear to converge to the same "limit". Earlier estimates of "this limit" involved far smaller clusters and actually referred to transcients.

A2) The metric properties of DLA are clearly not self similar. DLA clusters become rounder as they grow.

The project's earlier results on this topic had been obtained in 1991-1992; they were reported by Mandelbrot at the Hamburg Fractals meeting (July '92) and published in

Physica A 191, 1992, 95-107. They pioneered an altogether new attack on the concept of lacunarity (see Part B), and suggested that, while fractal dimension remains constant during growth, the cluster's lacunarity decreases with no limit in sight. This far-reaching conclusion had to be tested by other means, including means that are more conventional therefore better understood.

Mandelbrot, Kaufman, Lam and Yekutieli have compared successive "shells" of the cluster, the k-th "λ-shell" being defined as the set of atoms between the $N\lambda^k$-th and the $N\lambda^{k+1}$-th, in the order they joined the cluster. One element of scale is the average distance σ_1 of a shell's atom to the origin. Other elements of scale are of the form

$\sigma_k = k$-th root of the k-th moment of the difference between the distance and σ_1.

If the cluster had been self similar, the ratios of the form σ_k/σ_1 would have been constant. In fact, they were found, for all k, decrease with no limit in sight. This finding was reported in the Budapest Fractals meeting (August 1993) and is to appear in the Proceedings.

Part B. Development of the concept of numerical lacunarities.

Two Cantor sets on the line can have identical dimensions, yet "look" completely different from each other: some are obviously fractal, but others "mimic" an interval or its end points. This observation was made in *The Fractal Geometry of Nature* and used in mainly qualitative ways to tune a fractal, so as to make it fit data concerning galaxies. Since then, lacunarity has become equally important to the study of DLA. During 1992-1993, scattered efforts toward quantitative measures of lacunarity was brought together. A brief survey is about to be published in a book titled *Fractals in Biology and Medicine.*

The basic idea is that fractals obey a large number of scaling relations of the form $Y = FX^E$. Both the prefactor F the exponent E carry information about the fractal. It is well known that the exponent E is the fractal dimensions D (or a function of D). The additional information summarized by E concerns one aspect of the loose notion of texture.

Stated in practical terms, the most obvious prefactor concerns the number of boxes in box counting.

Stated in mathematical terms, the most recommended numerical measure of lacunarity is the ratio

Hausdorff measure/Minkowski contents.

The Minkowski contents is uniquely defined (except for overly regular cases in which one must distinguish a "sup" and an "inf" version). Lacunarity changes the Minkowski content from being an esoteric mathematical curiosity to being a physical notion.

An alternative already mentioned in *The Fractal Geometry of Nature* as possible measure of lacunarity is, mathematically, the Besicovitch density, more precisely, it is some average of the upper and the lower density. The value of the unavoidable difference between these densities is also of interest. Then various quantities have now been evaluated for a variety of interesting examples.

The reactions between the above two measures are delicate and are in the process of being investigated.

Indirectly related to these topics is a papaer by D. Stauffer, A. Aharony and B. Mandelbrot in *Physica A 196, 1993, 1-5* . It evaluates numerically the lacunarity of Brownian motion in 3 and 4 dimension and compares it with the known mathematical value. They fit.